Lesson Description

In this lesson we:

- Sketch the graphs of cubic functions in the standard form \( f(x) = ax^3 + bx^2 + cx + d \).
- Find the equation of a cubic function.
- Work with the graph of the derivative function.

Summary

The cubic graph has the general equation \( f(x) = ax^3 + bx^2 + cx + d \).

The cubic function can take on one of the following shapes depending on whether the value of \( a \) is positive or negative:

If \( a > 0 \)  

If \( a < 0 \)

Rules for Sketching the Graphs of Cubic Functions

Intercepts with the Axes

- For the y-intercept, let \( x=0 \) and solve for \( y \).
- For the x-intercept(s), let \( y=0 \) and solve for \( x \).

Stationary Points

- Determine \( f'(x) \), equate it to zero and solve for \( x \).
- Substitute the \( x \)-values of the stationary points into the original equation to obtain the corresponding \( y \) values.
- If the function has two stationary points, establish whether they are maximum or minimum turning points by referring to the shape.

Points of Inflection

- If the cubic function has only one stationary point, this will be a point of inflection that is also a stationary point.
- For points of inflection that are not stationary points, find the second derivative and equate it to 0 and solve for \( x \).
Rules for Finding the Equation of a Given Graph

- If the 3 x-intercepts of the graph are known, start with the equation
  \[ y = a(x - x_1)(x - x_2)(x - x_3) \]
- If the graph has a turning point on one of the x-intercepts, use the equation
  \[ y = a(x - x_1)(x - x_2)^2 \]
- If the turning points are known, substitute into \( f''(x) = 0 \)

Test Yourself

Question 1
The gradient of the curve of \( y = x^3 + ax^2 \) is equal to \(-4\) at the point \( x = 2 \). The value of \( a = \ldots \)
A. 4  
B. \(-4\)  
C. \(-3\)  
D. 2

Question 2
What is the value of \( y \) on the graph \( y = 3x^2 - 2x + 1 \) where the gradient is 4?
A. 1  
B. 22  
C. 2  
D. 41

Question 3
Determine \( f'(x) \) if \( f(x) = \frac{4}{x} \)
A. 4  
B. \(4x^{-1}\)  
C. \(-4x\)  
D. \(\frac{-4}{x^2}\)

Question 4
P(2 ; 4) is a point on the curve \( y = x^2 \). The gradient at this point is:
A. 2  
B. 4  
C. 8  
D. 0
Question 5
Determine \( f'(-1) \) if \( f(x) = 2x^3 - 3x^2 + x - 1 \)
A. 23  
B. -25  
C. 13  
D. -29

Question 6
Given the function \( f(x) = -x^3 - x^2 + 5x - 3 \). The x-intercepts are:
A. (3;0) and (1;0)  
B. (-3;0), (1;0) and (-1;0)  
C. (-3;0) and (1;0)  
D. (3;0), (1;0) and (-1;0)

Question 7
The point of inflection of the function \( y = -2x^3 - 12x^2 - 18x \) is:
A. (2 ; 4)  
B. (4 ; -2)  
C. (2 ; 0)  
D. (-2 ; 4)

Question 8
At which point does the curve of \( y = x^3 - x^2 - x - 8 \) have a local minimum?
A. (1 ; 0)  
B. (0 ; -8)  
C. \( \left( \frac{-1}{3} ; 0 \right) \)  
D. (1 ; -9)

Question 9
The curve below represents the graph of \( y = ax^3 + bx^2 + cx + d \). The values of \( a \), \( b \), \( c \) and \( d \) are:
A. \( a = \frac{1}{2}; b = -2; c = \frac{-3}{2}; d = 9 \)  
B. \( a = \frac{1}{3}; b = 2; c = \frac{-3}{2}; d = -9 \)  
C. \( a = \frac{-3}{2}; b = \frac{-1}{2}; c = -2; d = 9 \)  
D. \( a = \frac{-3}{2}; b = 2; c = \frac{3}{2}; d = -9 \)
Question 10

The derivative function of a cubic graph will be:

A. Cubic
B. Exponential
C. Linear
D. Quadratic

**Improve your Skills**

**Question 1**

Given: \( f(x) = -2x^3 + 5x^2 + 4x - 3 \). Draw a neat sketch of \( f(x) \). Clearly indicate the intercepts with the axes, as well as the coordinates of the turning and inflection point(s).

**Question 2**

Given: \( f(x) = x^3 - 3x - 2 \). Draw a neat sketch of \( f(x) \). Clearly indicate the intercepts with the axes, as well as the coordinates of the turning and inflection point(s).

**Question 3**

Given: \( g(x) = x^3 + 8 \). Draw a neat sketch of \( f(x) \). Clearly indicate the intercepts with the axes, as well as the coordinates of the turning and inflection point(s).

**Question 4**

Sketch the graph of the cubic function \( f(x) \) that is continuous for all \( x \) and has the following properties:

\[
\begin{align*}
f(-3) &= 4, & f'(-3) &= 0, & f''(-3) &> 0, & f(1) &= 32, & f'(1) &= 0, & f''(1) &< 0
\end{align*}
\]

**Question 5**

Determine the equation of the following graph in the form \( y = ax^3 + bx^2 + cx + d \).
Question 6

The sketch represents the graph of \( f(x) = ax^3 + 5x^2 + 4x + b \). \( E \) is the point (2 ; 9) and the coordinates of \( D(0 ; -3) \) and \( A(-1 ; 0) \). \( E \) is a local maximum.

Prove that \( a = -2 \) and \( b = -3 \)

Question 7

The diagram below represents the graph of \( y = f'(x) \), which is the graph of the derivative of the cubic function \( f(x) \).

7.1) What is the gradient of the tangent to the graph of \( f(x) \) at \( x = 0 \)?

7.2) For which value of \( x \) will there be a tangent to the curve of \( f \) which will be parallel to the tangent in 7.1?

7.3) For which value of \( x \) will \( f(x) \) be decreasing?

7.4) Write down the \( x \) co-ordinates of the turning points of \( f \) and state whether they are local maximum or minimum turning points.

7.5) If it is further given that the \( x \)-intercepts of the graph of \( f \) are -2, 2 and 7, use the information at your disposal and draw the graph of \( f \). It is unnecessary to determine the \( y \) – coordinates of the turning points.